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Non-polynomial solutions to the q-difference form of the Harper equation

C Micu and E Papp†

Department of Physics, Str. Victoriei nr. 76, University of Baia Mare, RO-4800 Baia Mare, Romania

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Abstract. This paper deals with the derivation of non-polynomial solutions to the q-difference form of the Harper equation. Only quasiclassical approximations proceeding this time to first and second orders are discussed. The exact non-polynomial zero-energy solution to the above q-difference equation has also been presented.

1. Introduction

A q-difference formulation of the Harper equation [1] has recently [2] been presented. One commences by incorporating the group of magnetic translations [3] into the symmetry structure of the quantum-group $sl_q(2)$ [4]. This q-difference equation looks like

$$\mathrm{i}\left(\frac{1}{z}+qz\right)\psi(qz)-\mathrm{i}\left(\frac{1}{z}+\frac{z}{q}\right)\psi(q^{-1}z)=E\psi(z) \tag{1}$$

where

$$q=q_H=\exp\left(\frac{\mathrm{i}}{2}\Phi\right)=\exp\left(\mathrm{i}\pi\frac{P}{Q}\right) \tag{2}$$

has the meaning of a pertinent q-deformation parameter. The magnetic field per plaquette is denoted by

$$\Phi=2\pi\frac{P}{Q} \tag{3}$$

where P and Q are mutually prime integers. Accordingly, q expresses roots of unity for which $q^{2Q}=1$, which means in turn that the pertinent $sl_q(2)$-representations are finite dimensional [4, 5]. Putting $z=q^{2n}$, where n is a real integer, leads after some steps to the usual discrete form of the Harper equation via $\psi_n=\psi(q^{2n})$ [2].

Thus (1) represents a q-difference approach to Bloch electrons on a two-dimensional lattice penetrated by a perpendicular magnetic field. Now the q-deformation is an inherent attribute of the physical description, like the $SU_q(2)$-symmetry of the Heisenberg XXZ spin chain [6, 7]. In general, the q-parameter exhibits real or imaginary values, so that the underlying 'classical' limit, as for example the XXX spin chain, is implied as $q\to 1$. Moreover, complex q-values have also been proposed, which leads to a nontrivial generalization [8, 9].

† E-mail address: epapp@univer.ubm.ro

So far, (1) has been solved in terms of the Bethe ansatz [2, 10]. The polynomial solutions obtained in this manner are quite valuable, but they are presented in an implicit form, as shown, for example, by (5), (6) and (9) in [2]. The explicit zero-energy polynomial solution has also been written down [11]. However, the derivation of further explicit solutions and/or approximations remains desirable. We shall then derive the non-polynomial counterpart of the zero-energy solution mentioned above, which represents the main contribution of this paper. For this purpose one begins by discussing related quasiclassical approximations.

2. Probing the quasiclassical description

The quasiclassical description of (1) provides useful insights into a better understanding of exact q-dependent solutions, such as the zero-energy one. Now one has $|q_H| = 1$, so that in order to perform the quasiclassical description we have to start from the approximation

$$q = q_H \cong 1 + \epsilon \tag{4}$$

working for $Q \gg \pi P$, in which one has

$$\epsilon = \mathrm{i}\frac{\pi P}{Q}\left(1 + \mathrm{i}\frac{\pi P}{2Q}\right) \tag{5}$$

to second P/Q-order. Using (4), we then have to realize that (1) leads to an infinite number of differential equations. Indeed, let us consider that

$$U \equiv -\mathrm{i}E = \sum_{k=1}^{\infty} \epsilon^k U_k \tag{6}$$

which definitely incorporates the present quasiclassical effects. It is also clear that now higher-order corrections to the wavefunction are not accounted for. One would then obtain

$$(1 + z^2)\psi'(z) + z\psi(z) = \tfrac{1}{2}U_1\psi(z) = -U_2\psi(z) \tag{7}$$

to first and second ϵ-orders, whereas

$$\tfrac{1}{3}z^2(1 + z^2)\psi'''(z) + z(1 + 2z^2)\psi''(z) + (1 + 3z^2)\psi'(z) + z\psi(z) = U_3\psi(z) \tag{8}$$

works to third ϵ-order. The primes denote differentiations with respect to z, as usual. Similar equations can also be written down to higher orders. Next it can easily be verified that (7) exhibits the solution

$$\psi(z) = \frac{C}{\sqrt{1 + z^2}} \exp\left(\frac{a}{2} \arctan z\right) \tag{9}$$

in which a is a parameter which remains to be established later in terms of suitable boundary condition. Then

$$U_1 = -2U_2 = a \tag{10}$$

so that

$$E = -\pi \frac{a\ P}{Q} + \mathrm{O}\left(\frac{P^3}{Q^3}\right). \tag{11}$$

Unfortunately, starting with (8), (9) does not fulfil higher-order equations. Thus, one becomes faced with a rather restricted quasiclassical description working to first and second ϵ-orders only. It could be emphasized that the accuracy of the quasiclassical description could eventually be enhanced by resorting to suitable modifications of (1), but such questions go beyond the immediate scope of this paper. In this general context, it should be noted, however, that in order to assure a positive norm, an extra term has to be inserted into (1), as shown recently [12].

Next we have to remark that imposing a boundary condition such as

$$\psi(1) = \exp(\gamma\pi)\psi(-1) \tag{12}$$

gives in general

$$a = 4(\gamma + n) \tag{13}$$

by virtue of (9), where γ is a real boundary parameter and where n denotes an arbitrary integer. Accordingly, the wavefunction (9) can be normalized within the interval $z \in [-1, 1]$. It should be remarked that in such a case the normalization integral remains unchanged under $a \rightarrow -a$. In particular, putting $\gamma = 0$ and $n = 0$ one obtains the zero-energy solution

$$\psi(z) = \psi_0(z) = \frac{C}{\sqrt{1+z^2}} = C\sum_{n=0}^{\infty}(-1)^n z^{2n}\frac{(2n-1)!!}{(2n)!!} \tag{14}$$

which exhibits an apparent non-polynomial form. Using the normalization condition

$$\int_{-1}^{+1} \psi_0^*(z)\psi(z)\,\mathrm{d}z = 1 \tag{15}$$

then gives $C = |C| = \sqrt{2/\pi}$, so that the above zero-energy wavefunction becomes well established.

So far it has been assumed that the z-coordinate exhibits real values. However, complex z-values on the unit circle such as

$$z = q^{2n} = \exp(2\mathrm{i}\pi nP/Q) \tag{16}$$

where $n = 0, 1, \ldots, Q$, should also be accounted for. Now we have to keep in mind the fact that the complex conjugation is given by $z^* = 1/z$. For convenience we shall then restrict ourselves to the zero-energy wavefunction (15), which will be rewritten equivalently as

$$\tilde{\psi}_0(z) = \frac{\tilde{C}}{\sqrt{1+z^2}}. \tag{17}$$

Proceeding in as close a manner as possible, we shall proceed by converting the input z-integral

$$I = \left|\int \tilde{\psi}_0^*(z)\tilde{\psi}_0(z)\,\mathrm{d}z\right| = |\tilde{C}|^2\left|\int \frac{z\,\mathrm{d}z}{1+z^2}\right| \geqslant 0 \tag{18}$$

into an integral over n, where the integration limits remain to be clarified later. Accounting for

$$\mathrm{d}z = 2\mathrm{i}\pi\frac{P}{Q}z\,\mathrm{d}n \tag{19}$$

and transforming the n-integral into a summation over n, gives the discretized norm

$$I \rightarrow I_{Q,P} = \pi\frac{P}{Q}|\tilde{C}|^2|S_{Q,P}| \tag{20}$$

where

$$S_{Q,P} = \sum_{n=0}^{Q}\frac{1}{\cos 2n\pi\frac{P}{Q}}. \tag{21}$$

Choosing, for example, $P = 1$ then yields $|S_{Q,1}| = 1$ if Q is an even number, whereas

$$|S_{Q,1}| = \begin{cases} Q+1 & (Q+1)/2 = \text{odd} \\ (Q+1)/2 & (Q+1)/2 = \text{even} \end{cases} \tag{22}$$

works for odd Q-numbers. This is a non-trivial result which can be viewed as being reminiscent to the hierarchical attributes characterizing the spectrum of the original Harper equation. In this latter respect it should be mentioned that besides the complex WKB method [13], other quasiclassical descriptions to the original discrete Harper-equation have been done [14–17]. A suggestive plot concerning the Q-dependence of $S_{Q,P}$ is presented in figure 1 for $Q \in [0, 17]$ and $P = 1$. This means that Q also stands for Q/P. One remarks the onset of a sequence of discretized scars growing specifically with Q, quite sharp maxima included.

Using (22) one would then obtain

$$\tilde{C} = \sqrt{\frac{2}{\pi}}\sqrt{\frac{Q}{Q+1}} \tag{23}$$

for even $(Q+1)/2$ values, so that $\tilde{C} \cong C$ if $Q \gg 1$. As a matter of fact, this latter $\tilde{C} \cong C$ solution is produced effectively by the modified norm

$$\langle\tilde{\psi}_0|\tilde{\psi}_0\rangle_{\text{mod}} = \frac{\pi}{\ln 2}\int_0^1 \tilde{\psi}_0^*(z)\tilde{\psi}_0(z)\,\mathrm{d}z \tag{24}$$

which may be useful for subsequent q-deformations. One proceeds in a similar manner in the other cases.

3. The derivation of the exact zero-energy solution

We are now ready to derive the exact zero-energy non-polynomial solution to (1). Using the symmetrized Jackson q-derivative (see, for example, [18–21])

$$\partial_q f(z) = \frac{f(qz) - f(q^{-1}z)}{z(q-q^{-1})} \tag{25}$$

one realizes immediately that (1) can be rewritten equivalently as

$$\partial_q\psi(z) + z\partial_q(z\psi(z)) = W\psi(z) \tag{26}$$

where now

$$E = \mathrm{i}\left(q - \frac{1}{q}\right)W = -2\sin\left(\pi\frac{P}{Q}\right)W. \tag{27}$$

It is also clear that

$$\partial_q z^n = [n]_q z^{n-1} \tag{28}$$

where the present quantum number reads

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}} = \frac{\sin(n\pi P/Q)}{\sin(\pi P/Q)}. \tag{29}$$

As a next step let us insert the (non-polynomial) power series expansion

$$\psi(z) = \psi_q(z) = \sum_{n=0}^{\infty} c_n z^n \tag{30}$$

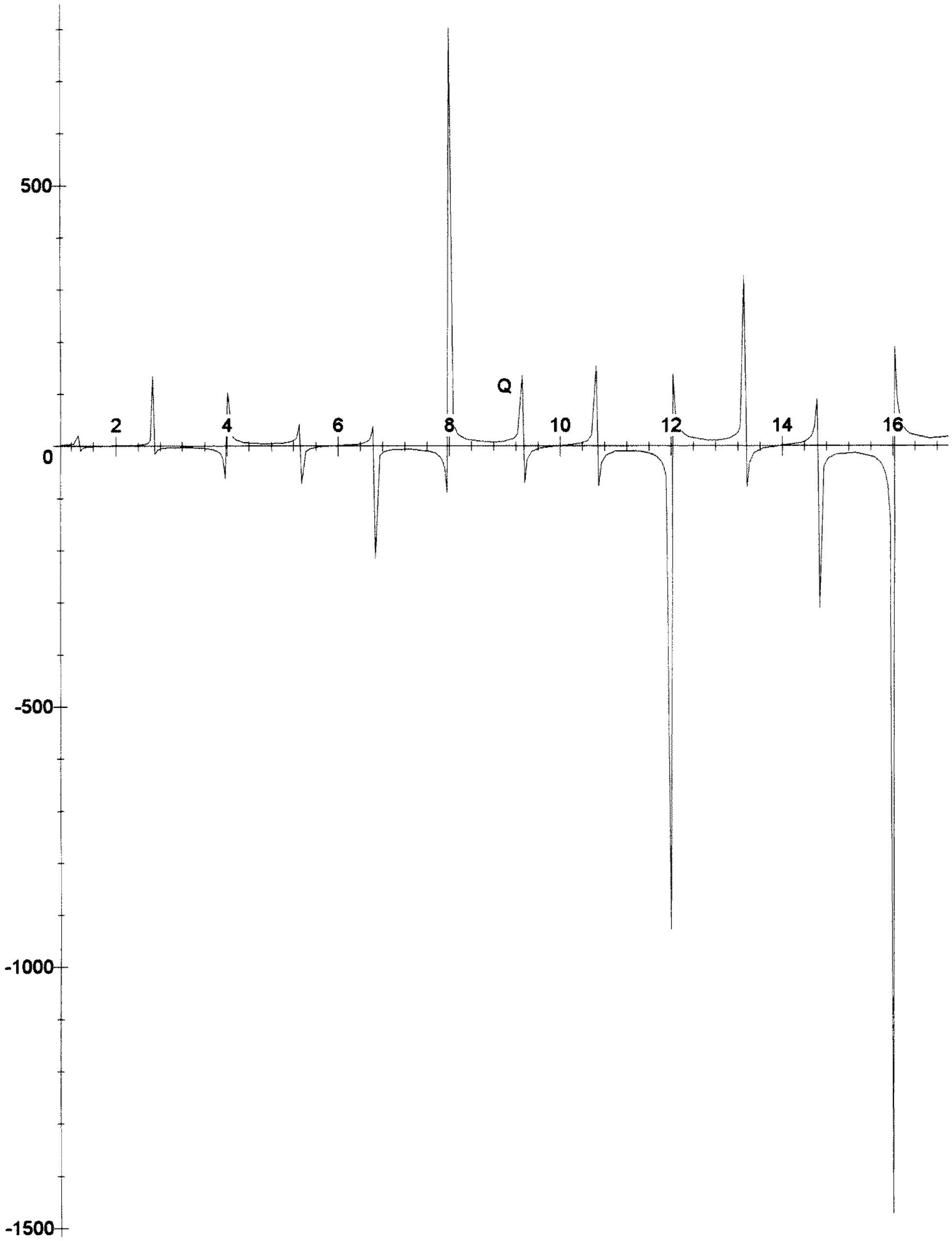

Figure 1. The Q dependence of $S_{Q,P}$ for $Q \in [0, 17]$ and $P = 1$.

into (26), where $c_0 \equiv c_0(q) \neq 0$. This yields the three-term recurrence relation

$$Wc_n = [n+1]_q c_{n+1} + [n]_q c_{n-1}. \tag{31}$$

Putting $W = 0$ leads to

$$c_{2n+1} = 0 \tag{32}$$

and

$$c_{2n} = (-1)^n \frac{[2n-1]_q!!}{[2n]_q!!} c_0 \tag{33}$$

where

$$[2n-1]_q!! = [1]_q[3]_q \ldots [2n-1]_q \tag{34}$$

and

$$[2n]_q!! = [2]_q[4]_q \ldots [2n]_q. \tag{35}$$

It is also understood that

$$[0]_q!! = [-1]_q!! = 1. \tag{36}$$

Thus, the zero-energy non-polynomial wavefunction is given solely by

$$\psi_q^{(0)}(z) = c_0(q) \sum_{n=0}^{\infty} (-1)^n \frac{[2n-1]_q!!}{[2n]_q!!} z^{2n} \tag{37}$$

which reproduces the zero-energy form of the 'classical' wavefunction (9) as $q \to 1$. Indeed, one has

$$\psi_1^{(0)}(z) = \frac{c_0(1)}{\sqrt{1+z^2}} = c_0(1) \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} z^{2n} \tag{38}$$

which reproduces precisely (9) as soon as $C = c_0(1)$. The absence of a second non-polynomial solution can also be understood in terms of the classical counterpart of (26), i.e. of (7), which is apparently a first-order differential equation. We argue that the $W \neq 0$ case can be treated in a similar manner, but this time the calculations are more involved. However, our expectation in this latter case is that $\psi_q(z)$ should be expressed in terms of a suitable q-deformation of (9).

The q-norm characterizing the q-deformed wavefunction (37) comes from the definite q-integral

$$\int_a^b \psi_q^{(0)*}(z)\psi_0^{(0)}(z)\, \mathrm{d}_q z = |c_0(q)|^2 \sum_{n,n'}^{\infty} (-1)^{n+n'} \frac{[2n-1]_q!![2n'-1]_q!!}{[2n]_q!![2n']_q!!} \times \frac{b^{2n-2n'+1} - a^{2n-2n'+1}}{[2n-2n'+1]_q} \tag{39}$$

where now the complex conjugation $z^* = 1/z$ has also been accounted for. What then remains is to choose suitable integration limits. Equation (24) would suggest the $z \in [0, 1]$ choice, but in this case one has some troubles with the power exponent $2n - 2n' + 1$ in (39), which is able to exhibit unfortunately negative values. We shall then resort to the q-deformation of equation (15), now for $z^* = 1/z$. Inserting the integration limits $a = -1$ and $b = 1$ then gives

$$b^{2n-2n'+1} = -a^{2n-2n'+1} = 1 \tag{40}$$

which enables us to say that the q-integral (39) is a convergent one. Equivalently, the q-integral relying on (25), i.e.

$$\int_0^c f(z)\, \mathrm{d}_q z = |qq^{-1}| \sum_{j=0}^{\infty} \frac{c}{q^{2j+1}} f\left(\frac{c}{q^{2j+1}}\right) \tag{41}$$

where c is an arbitrary real number, can also be applied. At this end it should also be noted that using (25) opens the way to the derivation of further explicit polynomial solutions, too [22].

4. Conclusions

In this paper we succeeded to establish alternative quasiclassical non-polynomial solutions to the q-difference form of the Harper equation (1). Proceeding quasiclassicaly we found the discretized norm (20), which exhibits a typical scar behaviour, as shown in figure 1. It is also clear that the boundary condition (12) reasonably fulfils necessary requirements, though other proposals may be conceivable. In addition, the exact non-polynomial zero-energy solution to (1) has been written down. The above results provide a deeper theoretical understanding of the quantum-mechanical capabilities of (1). We can then say that the q-difference form of the Harper equation exhibits rich structures going beyond the Bethe ansatz solution discussed before [2].

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